The plunge mode, mode 2, exists up to a speed of 186.5 ft/s. Toward the end of this range, the frequency decreases sharply. It then meets up with an additional oscillatory mode, mode 3. The frequency of mode 3 decreases further with reducing speed, becoming zero at a speed of 145.5 ft/s. Note that mode 3 has a corresponding complex conjugate eigenvalue that is not a valid root because its frequency does not match the assumed k value. As the frequency of mode 3 reaches zero, the complex conjugate eigenvalue becomes a valid root. As the speed increases from 145.5 ft/s, one real root, mode 4, becomes less stable and becomes unstable at the divergence speed, 216.5 ft/s. The other real root, mode 5, becomes more stable with increasing speed.

Discussion

It can be seen from Fig. 1 that the method of successive approximation would not converge to the second root of the second eigenvalue. The plot of imaginary part of the second eigenvalue p_2 crosses the line $2kV/c$ from below, implying that the slope of $\text{Im}(p_2)$ with respect to $2kV/c$ is greater than 1. Note that a Newton–Raphson solution method[8] would still converge.

The plot of the imaginary part of the second eigenvalue (Fig. 1) retains its shape, but moves to the right with increasing speed and to the left with decreasing speed. This is consistent with the appearance of the two real roots and one oscillatory root at 145.5 ft/s and the disappearance of the two oscillatory roots at 186.5 ft/s. Note that the shape of the plot is peculiar to Rodden, Harder, and Bellinger's form[6] of the p–k flutter equation.

From Fig. 2, it can be seen that a mode tracking procedure would track the plunge mode up to the speed where the mode ceases to exist or turns around and then converge to one of the real roots. This is similar to the behavior reported in Ref. 4, except that the solution converged to the real root at speeds where the oscillatory root should still exist.

Summary

Whenever the frequency of a structural mode goes to zero, one would expect the complex root to be replaced by two real roots at higher speeds. A mode tracking procedure would track only one of the real roots, implying that the solution would be incomplete. The divergence roots in the case of the two examples of Ref. 4 are the logical continuation of structural modes after their frequencies have gone to zero. Calling them aerodynamic lag roots does not seem justified.

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Unrestrained Aeroelastic Divergence and the *p*–*k* Flutter Equation

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Nomenclature

B	=	viscous structural damping matrix
c	=	airfoil chord
g	=	structural damping coefficient, also $2\text{Re}(p)c/[\ln(2)V]$
K	=	stiffness matrix
k	=	reduced frequency, $\omega c/2V$
M	=	inertia matrix
m	=	airfoil mass per unit span
p	=	differential operator, $\mathrm{d}/\mathrm{d}t$
Q	=	matrix of generalized aerodynamic forces, a function of Mach number and $pc/2V$
Q^I	=	imaginary part of Q
Q^R	=	real part of Q
$\{u\}$	=	vector of degrees of freedom
V	=	true airspeed
μ	=	airfoil mass ratio $m/[\pi\rho(c/2)^2]$
ρ	=	air density
ω	=	angular frequency, $\text{Im}(p)$

Introduction

IN Ref. 1 the aeroelastic divergence of two unrestrained airfoil–body systems was investigated using the British (p–k) flutter method. It was concluded that the systems diverged in an oscillatory fashion rather than quasi statically as is the case with restrained systems. It was also found that the divergence speeds as determined from the flutter analysis were slightly different from those obtained by quasi-static unrestrained divergence analysis.[2]

In the present study, the second example of Ref. 1 was analyzed using four forms of the p–k flutter equation, namely, Hassig's form,[3] Hassig's form with the rigid plunge displacement degree of freedom eliminated, Rodden, Harder, and Bellinger's form,[4] and the exact equation of motion. These results show the link between quasi-static unrestrained divergence analysis and the dynamic stability methods.

Solutions

The characteristics of the two examples of Ref. 1 are the same except for the center of gravity location: In example 1 it is at 37% chord and in example 2 it is at 45% chord. The chord is 6 ft, the elastic axis is at 40% chord, the radius of gyration about the elastic axis is 25% of the chord, and the mass ratio $\mu = 20.0$. The uncoupled bending and torsion frequencies are 10.0 and 25.0 rad/s, respectively, with equal structural damping coefficients $g = 0.03$ in both modes. The airfoil plunge spring is attached to a body with only a plunge degree of freedom and mass equal to the airfoil mass.

Incompressible flow was assumed, and Jones's approximation[5] to the Theodorsen circulation function was used in the calculation of the aerodynamic coefficients. A root search technique[6] was used rather than the traditional mode tracking method of solution. The damping values were all normalized as defined in Ref. 7 for nonoscillatory roots. This was necessary to show smooth transitions from complex to real roots.

The equation of motion of the system is

$$\left[Mp^2 + Bp + K - \tfrac{1}{2}\rho V^2 Q(pc/2V)\right]\{u\} = 0 \tag{1}$$

For most practical problems, the solution of Eq. (1) is a formidable task, mainly because of the dependence of the generalized forces

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on p. The generalized forces are usually only computed for harmonic motion, that is, $\mathrm{Re}(p) = 0$. However, if $Q(pc/2V)$ is available as an analytical function for complex p, Eq. (1) can be solved with reasonable effort.

In Hassig's form of the flutter equation,[3] it is assumed that the aerodynamic coefficients for harmonic motion is also valid for slowly growing or decaying oscillatory motion leading to the equation of motion

$$\left[Mp^2 + Bp + K - \tfrac{1}{2}\rho V^2 Q(ik)\right]\{u\} = 0 \qquad (2)$$

Rodden, Harder, and Bellinger[4] divided the matrix of generalized aerodynamic forces into an aerodynamic stiffness matrix and an aerodynamic damping matrix, leading to the equation of motion

$$\left[Mp^2 + \left(B - \tfrac{1}{4}\rho c V Q^I / k\right)p + \left(K - \tfrac{1}{2}\rho V^2 Q^R\right)\right]\{u\} = 0 \qquad (3)$$

The equations of motion are written in state-space form with order six and solved using a standard eigenvalue routine. The solution of Hassig's form of the flutter equation[3] is shown in Fig. 1 and consists of the elastic pitch and plunge modes (modes 1 and 2), the rigid plunge mode (mode 3), as well as the real roots associated with divergence (modes 4 and 5). The pitch mode flutters at 158.4 ft/s, and the rigid plunge mode flutters at 215.0 ft/s, whereas the elastic plunge mode remains stable. The divergence speed of 200.4 ft/s is exactly the speed predicted by quasi-static unrestrained divergence analysis.

The rigid plunge displacement degree of freedom can be eliminated from the equations of motion by defining aerodynamic coefficients relating lift and moment to plunge velocity rather than plunge displacement. The order of the problem is reduced to five, and the solution is shown in Fig. 2. The rigid plunge mode (mode 3) is now stable and nonoscillatory. An oscillatory root (mode 4) bifurcates from the rigid plunge mode at 148 ft/s, flutters at 215.0 ft/s, and joins one of the divergence roots at 260 ft/s. The divergence roots (modes 5 and 6) appear at 237 ft/s with a positive divergence rate $g = 0.86$.

The solution of Rodden, Harder, and Bellinger's form of the flutter equation[4] is shown in Fig. 3 and consists of the pitch mode

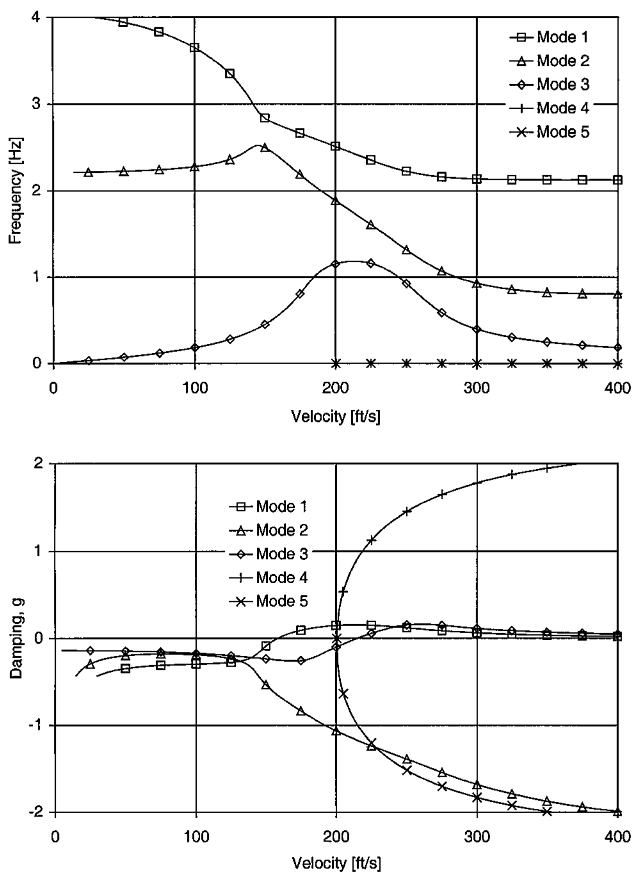

Fig. 1 Solution of Hassig's form[3] of the flutter equation with order six.

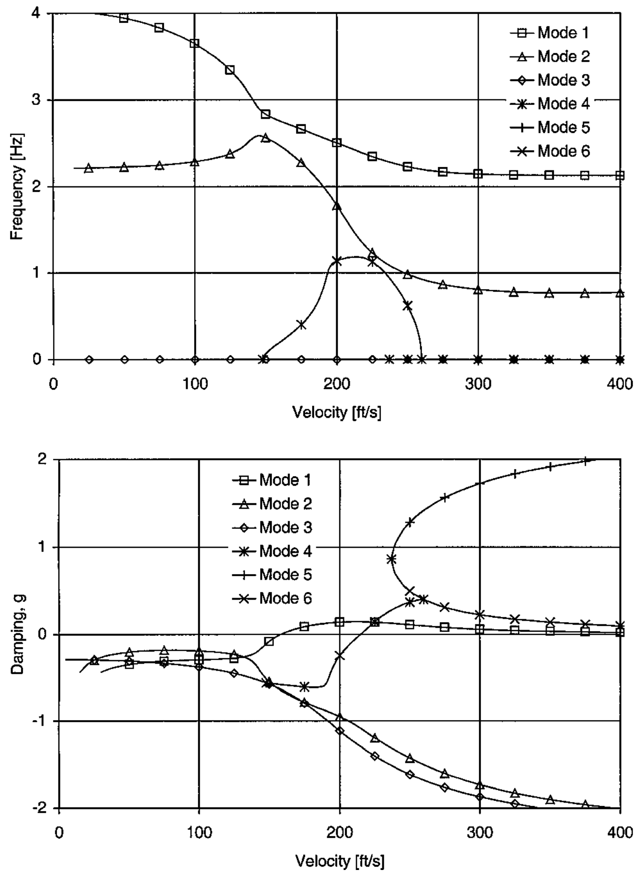

Fig. 2 Solution of Hassig's form[3] of the flutter equation with order five.

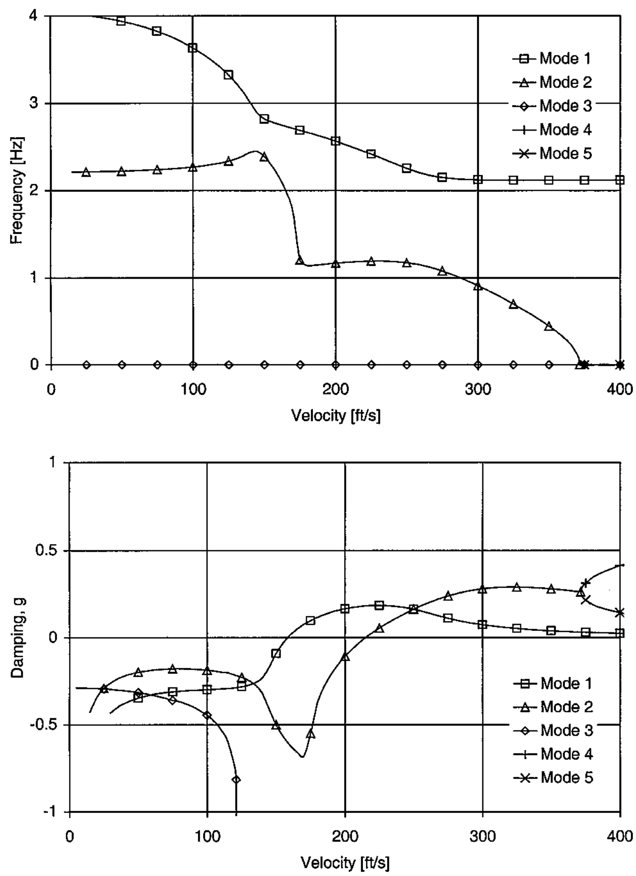

Fig. 3 Solution of Rodden, Harder, and Bellinger's form[4] of the flutter equation.

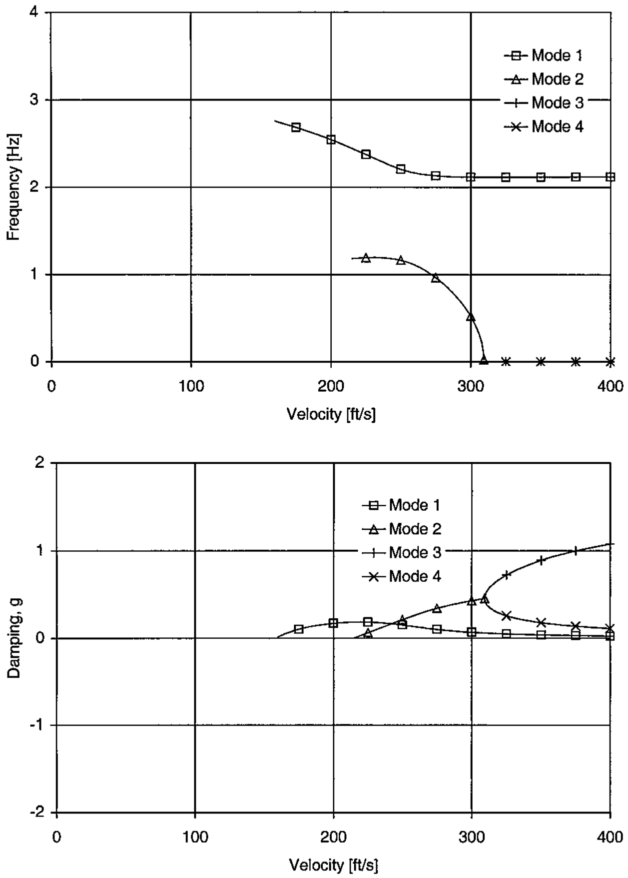

Fig. 4 Unstable roots of the exact equation of motion.

(mode 1), the elastic plunge mode (mode 2), and the rigid plunge mode (mode 3). The pitch mode flutters at 158.4 ft/s, and the elastic plunge mode flutters at 215.0 ft/s. The frequency of the elastic plunge mode goes to zero between 371 and 372 ft/s with a positive divergence rate $g = 0.26$. The mode then splits into two real roots, one becoming more unstable and the other becoming less unstable.

The unstable roots of Eq. (1) are shown in Fig. 4. The oscillatory roots were found by a mode tracking procedure starting at the flutter points. The real roots were found by a root search procedure with the aerodynamic matrix calculated for exponentially diverging motion. The solution is qualitatively similar to the solution of Rodden, Harder, and Bellinger's form of the flutter equation[4]; however, the frequency of the plunge mode goes to zero between 309.3 and 309.4 ft/s and the initial divergence rate $g = 0.46$.

It was verified that the divergence roots of Hassig's form[3] with order six (Fig. 1) and five (Fig. 2) migrate smoothly to the exact result (Fig. 4) by substituting $Q(fpc/2U)$ for $Q(pc/2U)$ in Eq. (1) and letting f vary from 0 to 1. The only surprising result was that the complete parabolic structure in Fig. 1 does not migrate, but first divides along the velocity axis. It was also verified that the solution of Eq. (1) with order six was identical to the solution with order five.

Discussion

In the solution of Hassig's form[3] of the flutter equation with order six the divergence roots appear independently of the other roots of the system. In the solution of Hassig's form with order five, the divergence roots are linked to the rigid plunge mode by a bifurcated root. In both these cases the instabilities at 158.4 and 215.0 ft/s would be regarded as flutter, whereas the origin of the real roots would be regarded as the onset of divergence.

In the solutions of Rodden, Harder, and Bellinger's form of the flutter equation[4] as well as the exact equation of motion, the real roots originate where the frequency of the elastic plunge mode goes to zero. It is open to interpretation whether the onset of divergence is where the root becomes unstable or where it becomes nonoscillatory. It is, however, not valid to compare the divergence speed predicted by the quasi-static unrestrained divergence analysis to the flutter point.

Summary

In Ref. 1 the accuracy of the quasi-steady unrestrained divergence analysis was questioned. Hassig's form[3] of the flutter equation with order six, which ignores the effect of exponentially decaying or growing motion on the aerodynamics, agrees exactly with the quasi-static unrestrained divergence analysis. This suggests that the flaw in quasi-static unrestrained divergence analysis is the assumption of quasi-static aerodynamics.

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Simplified Treatment of Unsteady Aerodynamics for Lifting Rotors

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Introduction

IT is sometimes useful in computing blade airloads to have approximate means to account for the nonstationary flow effects resulting from the complex time varying flowfields in which lifting rotors operate. This Note suggests such an approximation and compares the results with solutions obtained by more rigorous methods for the following two cases: 1) a two-dimensional oscillating airfoil with a shed wake remaining in the plane of the airfoil and 2) a two-dimensional oscillating airfoil including a returning shed wake located below the airfoil.

Reference 1 has formed the basis of most nonstationary flow aerodynamic analyses in which the wake may be assumed to remain in the plane of the airfoil. Reference 2 demonstrated the importance of considering the returning wake for rotors in vertical flight. It was shown that the blade damping could be reduced to very low values at integers of the ratio of blade frequency to the forcing frequency and could approach zero under conditions of no inflow. Both of these treatments are frequency based. It has been recognized for some time that, for rotors, a direct time-dependent treatment could be advantageous, as discussed in Ref. 3. This is particularly so in view of the highly variable changes in load, both temporal and spatial, associated with such a phenomenon as blade and vortex interaction, a primary contributor to the higher harmonic blade loadings of cruising flight.

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